Non-Boussinesq Treatment in the Posiedon Ocean Model

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1 Introduction

This document started out as a collection of notes about various terms in the conversion of Poseidon – a working notebook, not a paper. As I cleaned things up a little, I have added comments and text that make it more of the beginnings of a paper.

The Poseidon ocean model (Schopf and Loughe, 1995) was originally formulated as a reduced-gravity, quasi-isopycnal ocean model with a generalized vertical coordinate. It was formulated as a Boussinesq, hydrostatic model, where the surface elevation was diagnosed from the condition that horizontal gradients of pressure must vanish in the ocean abyss. The addition of a barotropic component has been made through a split-grid, split-explicit barotropic mode treatment that solves for the depth-integrated flow with short explicit timesteps coupled to slower "shear" or "internal" dynamics (Schopf, 1999).

The basic technique of making the split is to separate the pressure into a depth-independent component which depends on the elevation of the surface of the ocean relative to a constant geopotential, plus an internal component due to internal variations in the ocean's density. The velocity field is likewise decomposed into a depth-averaged component and a shear component. In the numerical integration of the equations of motion, the shear components are viewed as changing slowly in comparison with the depth-averaged flow, so the task of integrating the external mode is roughly aking to solving a barotropic model, but with a few additional slowly varying "forcing" terms that arise from the vertical averaging of the internal mode equations. Because of the layered nature of the model, the obvious choice for the surface pressure would be to write

$$\frac{\partial \eta}{\partial t} = \int \frac{\partial \tilde{h}}{\partial t} d\zeta \tag{1}$$

where η is the surface height, h is the thickness of each layer, and the integration is taken over the full range of layer coordinate (ζ).

In a Boussinesq formulation, this equation for surface height evolution fails to account for thermosteric effects. Since volume is conserved in the global integral, there is no direct change in surface height that can be accounted as due to warming or cooling of the water column. Given that the choice of h (the geometric thickness of each layer) is an arbitrary choice of a metric for the vertical coordinate transformation, there should be no inherent problem with employing it. The problem that comes, however, is in making the Boussinesq approximation, which states that

$$\frac{\partial \tilde{h}}{\partial t} = -\nabla \cdot \mathbf{v}\tilde{h}, \tag{2}$$

among other things. In fact, if temperature and salinity are being changed by the flow as well, $\tilde{h}$ will change due to warming or freshening (or both), although in an apparently small way. While the changes in thickness of each layer are completely dominated by (2), the problem really emerges during the solution of (1). Here, the dominant balance is that the flow is largely non-divergent, and the terms due to warming and cooling are now much more prominent in the balance.

A choice can be made between utilizing the Boussinesq formulation for the internal mode equations, but adding a term to the external mode equation which accounts for the thermosteric terms, or the abandonment of the Boussinesq approximation at the outset. As will be seen below, in the layered system envisioned here, the Boussinesq approximation can be dropped with virtually no computational penalty, and reqults in a clean simple system of equations very simplar in spirit to the original.

2 A Non-Boussinesq Treatment

Instead of defining a vertical scale factor

$$\tilde{h} = \frac{\partial z}{\partial \zeta}, \tag{3}$$

and buoyancy

$$b = g\frac{\rho_o - \rho}{\rho_o} \tag{4}$$

consider a hydrostatic model where the layers are described in pressure coordinates, so that

$$h = -\frac{1}{g\rho_o}\frac{\partial p}{\partial \zeta}. \tag{5}$$

The term h has units of meters, and is quite close to $\tilde{h}$ to the same accuracy that water at 1 decibar is 1 meter below the surface. In such a model, h represents the mass of seawater per unit of ζ. Instead of defining the buoyancy b, define the specific volume anomaly α

$$\rho^{-1} = \rho_o^{-1}(1+\alpha) \tag{6}$$

the variable h is related to $\partial z/\partial \zeta$ by

$$\frac{\partial z}{\partial \zeta} = h(1+\alpha). \tag{7}$$

The equations for the tendencies in mass, heat, salt, and momentum are now expressed in terms of h, T, S,and $\mathbf{v}$:

$$\frac{dh}{dt} = 0 \tag{8}$$

$$\frac{dhT}{dt} = \frac{1}{\rho_o C_p}\boldsymbol{\nabla}\cdot\mathbf{F} \tag{9}$$

$$\frac{dhS}{dt} = 0 \tag{10}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + (2\Omega+\omega)\times\rho\mathbf{v} = -\nabla_z p + \frac{\partial \tau}{\partial \zeta} - \rho\boldsymbol{\nabla}_\zeta K \tag{11}$$

where q is a vertical heat flux, τ is a vertical momentum flux, ω is the relative vorticity and K is the kinetic energy per unit mass $\mathbf{v}\cdot\mathbf{v}/2$.

The total derivative is gievn by

$$\frac{dhX}{dt} = \frac{\partial hX}{\partial t} + \boldsymbol{\nabla}_\zeta \cdot (\mathbf{v}hX) + \frac{\partial w_e X}{\partial \zeta}. \tag{12}$$

Finally, w_e is the cross-coordinate mass flux (similarly scaled by $g\rho_o$), and the subscript ζ on the divergence operator above denotes gradients taken along the coordinate surfaces, while the subscript z on the pressure gradient indicates that the gradient is taken along surfaces of constant geopotential (the conversion to a gradient along ζ follows). The evolution of α is diagnosed from the evolution of T and S and the equation of state. Conservation of mass is established by (8) and conservation of heat and salt by (9) and (10). The specific volume anomaly is not conserved.

Using $\rho_o = \rho(1+\alpha)$, the momentum equation can be multiplied by h/ρ to obtain

$$h\frac{\partial \mathbf{v}}{\partial t} + (2\Omega+\omega)\times h\mathbf{v} = -\frac{h(1+\alpha)}{\rho_o}\boldsymbol{\nabla}_z p + \frac{1}{\rho}\frac{\partial \tau}{\partial \zeta} - h\boldsymbol{\nabla}_\zeta K - w_e\frac{\partial \mathbf{v}}{\partial \zeta} \tag{13}$$

The choice of h over $\tilde{h}$ ensures that the equations are non-Boussinesq, and mass – not volume – is conserved.

2.1 Pressure Force

The pressure term needs further elaboration. The horizontal component of the pressure gradient is given by

$$\nabla_z p = \nabla_\zeta p - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial z}\nabla_\zeta z \tag{14}$$

$$= \nabla_\zeta p + \frac{g\rho_o}{1+\alpha}\nabla_\zeta z \tag{15}$$

Denoting the pressure term in (13) as P,

$$P = -h(1+\alpha)\nabla_z p/\rho_o = -h(1+\alpha)\nabla_\zeta p/\rho_o - gh\nabla_\zeta z \tag{16}$$

Noting (5) and (7), we have

$$\rho_o P = -\frac{\partial z}{\partial \zeta}\nabla_\zeta p + \frac{\partial p}{\partial \zeta}\nabla_\zeta z \tag{17}$$

For the zonal and meridional components of the pressure force, this relationship describes the pressure force in the form of a Jacobian, and can be expressed in the form proposed by Song(1999) or Lin (1997), among others.

The above relation can be expanded for the purpose of examining the potential to kinetic energy converison properties of the scheme.

The surface boundary condition is that pressure is continuous at $z = \eta$. For the purposes of the ocean model, the atmospheric pressure P_o is considered an external boundary condition, and the hydrostatic approximation is made within the ocean column. Thus one choise for expanding the equations could be

$$\begin{aligned} P = & -gh\nabla\eta - h(1+\alpha)\nabla P_o/\rho_o \\ & -gh(1+\alpha)\nabla\int_\zeta^0 h d\zeta' \\ & +gh\nabla\int_\zeta^0 h(1+\alpha)d\zeta' \end{aligned} \tag{18}$$

Removing the common terms, we have

$$\begin{aligned} P = & -gh\nabla\eta - h(1+\alpha)\nabla P_o/\rho_o \\ & -gh\alpha\nabla\int_\zeta^0 h d\zeta' + gh\nabla\int_\zeta^0 h\alpha d\zeta' \end{aligned} \tag{19}$$

In the remainder of this paper, the surface atmospheric pressure term will be considered insignificant in comparison with η.

3 Splitting the External Mode

The solution of the external mode as a separate equation (either through the use of a stream-function or the solution using split-explicit or implicit techniques) is done by integrating the equations of motion over the water column and splitting the flow into a depth-averaged motion and a shear. For this the vertical average is taken in a mass-weighted fashion:

$$\mathbf{V}_e = \int_{\zeta_b}^{0} h\mathbf{v}d\zeta/H \tag{20}$$

where H is the vertical integral mass:

$$H = \int_{\zeta_b}^{0} hd\zeta. \tag{21}$$

The shear flow $\mathbf{v}'$ is the departure of the velocity from $\mathbf{V}_e$.

The equation for the evolution of the depth-averaged flow is obtained by taking the vertical integral of (11) together with the vertical integral of $\mathbf{v}$ times (8). For evaluating the right hand side of the depth integrated momentum tendency, we need the depth integral of the pressure term:

$$\int_{\zeta_b}^{0} Pd\zeta = -gH\boldsymbol{\nabla}\eta + g\int_{\zeta_b}^{0} h\alpha\boldsymbol{\nabla}\int_{\zeta}^{0} hd\zeta'd\zeta - g\int_{\zeta_b}^{0} h\boldsymbol{\nabla}\int_{\zeta}^{0} h\alpha d\zeta'd\zeta \tag{22}$$

Note that if α is independent of depth, the only term in the pressure gradient is $g\nabla\eta$, as expected.

The pressure term can be separted into that part due to the surface elevation change, and that part due to internal baroclinic re-adjustment,

$$\int_{\zeta_b}^{0} Pd\zeta = gH\boldsymbol{\nabla}\eta + P_i \tag{23}$$

where P_i are the baroclinic terms, we note that the depth integrated equations are

$$\frac{\partial \mathbf{V}_e}{\partial t} = -g\nabla\eta - P_i/H + \ldots \tag{24}$$

$$\frac{\partial H}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{V}_e H \tag{25}$$

We can define a vertical average of α, such that the geometric change in depth over the water column is $H(1+\overline{\alpha})$ to obtain

$$\eta = H(1+\overline{\alpha}) + z_b \tag{26}$$

where $\overline{\alpha}$ is given by

$$H\overline{\alpha} = \int_{\zeta_b}^{0} h\,\alpha\,d\zeta. \tag{27}$$

In the split between fast and slow portions of the equations, the consistency requirement is that all of the vertically integrated mass convergence be accomplished by the fast mode terms. In integrating from "slow" step $t - \Delta t$ to $t + \Delta t$, the vertical integral of the mass flux used in any calculations should equal the time integral of $\mathbf{V}_e H$ made over the same interval. The conservation of total mass is thereby enforced via the differencing scheme for the "fast" integration, while the conservation of heat and salt are enforced via the "slow" steps. Because the surface height is diagnosed in (26) during the fast steps, there is no special consistency requirement for η.

The external mode equations that we solve are (24) and (25), where η is diagnosed from H, but for which $\overline{\alpha}$ is assumed to be held constant over the time period represented by one "slow" step.

4 JEBAR

The Joint Effect of Baroclinicity and Relief (JEBAR) appears in these equations when the curl of the vertically averaged momentum equations is taken (Sarkisyan and Ivanov, 1971; Mertz and Wright, 1992). Two effects are of interest: The accuracy of the JEBAR term itself and the demonstration that the pressure gradient force is irrotational over topography.

4.1 Pressure Torque Consistency

As discussed in Arakawa and Suarez (1983), the vertically integrated pressure trems must not generate momentum when integrated around a contour of constant bottom pressure.

Recall that the pressure gradient force is the combination of a total differential ($g\nabla\eta$) and an internal component P_i:

$$\mathbf{P}_i = g\int_{\zeta_b}^{0} h\alpha \left(\boldsymbol{\nabla}\int_{\zeta}^{0} hd\zeta'\right) - gh\left(\boldsymbol{\nabla}\int_{\zeta}^{0} h\alpha d\zeta'\right) d\zeta'' \tag{28}$$

Now let

$$\xi(\zeta) = -\int_{\zeta}^{0} h\alpha d\zeta' \tag{29}$$

$$\mu(\zeta) = \int_{\zeta}^{0} hd\zeta' \tag{30}$$

so we have

$$\mathbf{P}_i = g\int_{\zeta_b}^{0} -\frac{\partial \xi}{\partial \zeta}\boldsymbol{\nabla}\mu + \frac{\partial \mu}{\partial \zeta}\boldsymbol{\nabla}\xi d\zeta' \tag{31}$$

$$= g\int_{\zeta_b}^{0} -\frac{\partial}{\partial \zeta}(\xi\boldsymbol{\nabla}\mu) + \xi\boldsymbol{\nabla}\frac{\partial \mu}{\partial \zeta} + \frac{\partial \mu}{\partial \zeta}\boldsymbol{\nabla}\xi d\zeta' \tag{32}$$

$$= -g\xi(\zeta_b)\boldsymbol{\nabla}\mu(\zeta_b) + g\int_{\zeta_b}^{0}\boldsymbol{\nabla}\left(\frac{\partial \mu}{\partial \zeta}\xi\right)d\zeta' \tag{33}$$

Replacing the expressions for μ and ξ, and noting that

$$\mu(\zeta_b) = H \tag{34}$$

$$\xi(\zeta_b) = -\overline{\alpha}H \tag{35}$$

we have

$$\mathbf{P}_i = \overline{\alpha}H\boldsymbol{\nabla}H - \boldsymbol{\nabla}\left(\int_{\zeta_b}^{0} h\int_{\zeta}^{0} h\alpha d\zeta' d\zeta''\right) \tag{36}$$

The consisency condition that is required is that the line integral of $P_i \cdot d\mathbf{s}$ vanishes along contours of constant H. The first term vanishes by the definition of the path and gradient (i.e., the integral is specifically taken in the direction for which $\nabla H = 0$), and the second term vanishes due to its total differential form.

4.2 JEBAR

The JEBAR term arises by taking the curl of $\mathbf{P}_i/H$:

$$\text{JEBAR} = \nabla\times\overline{\alpha}\boldsymbol{\nabla}H - \nabla\times H^{-1}\int_{\zeta_b}^{0}\boldsymbol{\nabla}\left(h\int_{\zeta}^{0} h\alpha d\zeta'\right)d\zeta'' \tag{37}$$

$$= \nabla\overline{\alpha}\times\boldsymbol{\nabla}H - \nabla H^{-1}\times\int_{\zeta_b}^{0}\boldsymbol{\nabla}\left(h\int_{\zeta}^{0} h\alpha d\zeta'\right)d\zeta'' \tag{38}$$

In a model with thermal or saline forcing and strong topography, the generation of JEBAR by surface buoyancy flux can be seen through the direct forcing of $\overline{\alpha}$ at an angle to the gradient in bottom topography (since gradients of Pi are largely dominated by gradients in depth).

A simple example of this effect lies with the ploeward flow of a western boundary current along a continental shelf. In the presence of a pole-to-equator equilibrium atmospheric temperature gradient, the advection of warm water poleward will ensure a large surface cooling in the north relative to the south. This generates $\partial\overline{\alpha}/\partial y$ along a coast which has $\partial H/\partial x$. For a flow obeying the simple linearized version of the nondivergent equations:

$$J(\psi, f/H) = \nabla \times \frac{\tau}{H} + \mathrm{JEBAR}, \tag{39}$$

the cooling to the north and deepening to the east will provide a JEBAR effect that will shift the flow to deeper water – thus enhancing separation of the western boundary current from the coast.

5 Energetics

For the non-Boussinesq model, the kinetic energy is

$$KE = \int_{\zeta_b}^{0} h\mathbf{v}\cdot\mathbf{v}d\zeta \tag{40}$$

The potential energy is

$$PE = \int_{\zeta_b}^{0} hzd\zeta \tag{41}$$

or

$$PE = \int_{\zeta_b}^{0} h(\eta - \int_{\zeta'}^{0} h(1+\alpha)d\zeta'')d\zeta' \tag{42}$$

Again using (30) and (30)

$$PE = \eta H + \int_{\zeta_b}^{0} \frac{\partial\mu}{\partial\zeta}\mu d\zeta' + \int_{\zeta_b}^{0} \frac{\partial\mu}{\partial\zeta}\xi d\zeta' \tag{43}$$

Or

$$PE = \eta H - \mu^2(\zeta_b)/2 + \int_{\zeta_b}^{0} \frac{\partial\mu}{\partial\zeta}\xi d\zeta' \tag{44}$$

$$= \eta H - \mu^2(\zeta_b)/2 + \int_{\zeta_b}^{0} \frac{\partial\mu\xi}{\partial\zeta}d\zeta' - \int_{\zeta_b}^{0} \mu\frac{\partial\xi}{\partial\zeta}d\zeta' \tag{45}$$

$$H(\eta - H/2) - \int_{\zeta_b}^{0} h\int_{\zeta}^{0} h\alpha d\zeta'' d\zeta' \tag{46}$$

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