**Comments on Exercises in Chapter 6**

**6.6**

Use equation (6.8). If you use unnormalized polynomials, the formula is

$$c_k\langle P_k, P_k\rangle = \langle f, P_k\rangle,$$

where $\langle P_k, P_k\rangle = 2/(2k+1)$.

For $f(t) = \mathrm{e}^t$, you do a lot of integration by parts.

The coefficients for the unnormalized polynomials are

$$c_0 = (\mathrm{e} - \mathrm{e}^{-1})/2$$
$$c_1 = 2\mathrm{e}^{-1}/(2/3)$$
$$c_2 = (\mathrm{e} - 7\mathrm{e}^{-1})/(2/5)$$
$$c_3 = (-5\mathrm{e} + 37\mathrm{e}^{-1})/(2/7)$$

I will use the normalized polynomials (instead of those in equation (6.13)). The coefficients for the normalized polynomials are

$$\mathrm{e} - \mathrm{e}^{-1},\ 2\mathrm{e}^{-1},\ \mathrm{e} - 7\mathrm{e}^{-1},\ -5\mathrm{e} + 37\mathrm{e}^{-1}.$$

(You can work the problem either way.)

Some R code to form a vector of normalized Legendre polynomials is

```
P0<-function(t){
   return(as.vector(rep(.5,length(t))))
   }
PC1<-function(t){
   return(cbind(P0(t),3*t/2))
   }
PC2<-function(t){
   return(cbind(PC1(t),5*(3*t^2-1)/4))
   }
PC3<-function(t){
   return(cbind(PC2(t),7*(5*t^3-3*t)/4))
   }
```

Now to approximate $f(t) = \mathrm{e}^t$, using 4 terms, we form the approximation $\hat{f}_3$. In R, this is

```
e1 <- exp(1)
em1 <- 1/e1
c <- c(e1-em1,2*em1,e1-7*em1,-5*e1+37*em1)
f3hat<-function(t,c){
    return(PC3(t)%*%c)
    }
```

**6.6(a)**

The plot is straightforward. The R code is

```
t<-((1:201)-101)/100
plot(t,exp(t),typ='l')
lines(t,f3hat(t,c),lty=2)
```

The plots are right on top of each other, so you cannot see the separate lines.

**6.6(b)**

The error at $t = 0$ is `exp(0)-f3hat(0,c)` $= 0.0037$.

**6.6(c)**

The integrated squared error is

$$
\begin{aligned}
\int_{-1}^{1} \left(f(t) - \hat{f}_3(t)\right)^2 \mathrm{d}t \quad &= \quad \int_{-1}^{1} (\mathrm{e}^t - \\
&\quad (\mathrm{e} - \mathrm{e}^{-1})/2 + \\
&\quad 3\mathrm{e}^{-1}t + \\
&\quad 5(\mathrm{e} - 7\mathrm{e}^{-1})(3t^2 - 1)/4 + \\
&\quad 7(-5\mathrm{e} + 37\mathrm{e}^{-1})(5t^3 - 3t)/4\big)^2 \mathrm{d}t \\
&= \quad 0.000022.
\end{aligned}
$$

**6.6(d)**

We can try to make the shape of the weight function more similar to that of the approximant. Over the range $[-1, 1]$, $(1 + t)$ is similar in shape to $\mathrm{e}^t$ but with a little convexity, so we might try the Jacobis with $\alpha = 0$ and $\beta$ between 1 and 2.

**6.7**

The mean squared error for an unbiased estimator, such as $\bar{y}$ for estimating the sample mean, is just the variance. We know the variance of $\bar{y}$ goes down as $1/n$; hence, the true value of $\alpha$ is $-1$. The bias of $s$ is order $-1/2$; the order of the variance, and, consequently, of the MSE is also $-1$.

**6.9**

It is important to note that the domain is finite: $[0, 1]$.

$$
\mathrm{MISE}(\hat{f}) \quad = \quad \int_0^1 \mathrm{E}((\hat{f}(t) - f)^2)\mathrm{d}t
$$

$$
\begin{aligned}
&= \int_0^1 \sigma^2 \mathrm{d}t \\
&= \sigma^2.
\end{aligned}
$$

Note, in general, if the domain is $[a, b]$, then $\mathrm{MISE}(\hat{f}) = (b - a)\sigma^2$.

**6.10**

We have

$$
\begin{aligned}
\mathcal{S}(\hat{f}) &= \int_{-1}^{1} \left(\hat{f}(t)\right)^2 \mathrm{d}t \\
&= 3.627
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}(\hat{f}) &= \int_{-1}^{1} \left(\hat{f}''(t)\right)^2 \mathrm{d}t \\
&= 3.049
\end{aligned}
$$